Binary correlations in random matrix spectra

1979 J. Phys. A: Math. Gen. 12 L83

(http://iopscience.iop.org/0305-4470/12/5/002)

J. Phys. A: Math. Gen., Vol. 12, No. 5, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Binary correlations in random matrix spectra†

A Pandey and J B French‡

Department of Physics and Astronomy, University of Rochester, Roche...r, NY 14627

Received 29 November 1978

Abstract. The spectrum for a Gaussian orthogonal ensemble of random matrices augmented by a pairing interaction, which has been recently given by Edwards and Jones and by Jones, Kosterlitz and Thouless, is derived in other ways. Several extensions are given and relationships to other problems of current interest discussed. Stress is laid on the importance of the dominance of binary Hamiltonian correlations in the moments which define the density and relevant correlation functions.

The eigenvalue spectrum for a Gaussian orthogonal ensemble (GOE) of large-dimensional random matrices H, augmented by a multiple of the pairing interaction K (which has $(d-1)$ zero eigenvalues and one unit eigenvalue), has been recently given by Edwards and Jones (1976) and by Jones *et al* (1978). In this letter we give alternative methods of solution (the first of which is similar in spirit to that of Jones *et al*), and some extensions, which follow directly from a simple principle used by Wigner (1955) in his original solution for the GOE itself, and a corresponding counting theorem. The point of doing this is to establish connections between this problem and a number of other spectral and fluctuation problems which are of current interest.

We have $H_\alpha = H + \alpha K$. A particularly simple derivation of the ensemble average and the ensemble variance of the 'ground-state' energy E (i.e. the energy which tends to α for large α) comes from Brillouin–Wigner perturbation theory using αK as the unperturbed Hamiltonian. Thus, in the K-diagonal representation (the results being independent of this choice), we encounter

$$\Lambda_r = \sum_{i_1,\ldots,i_{r+1}\neq 1} H_{1i_1}H_{i_1i_2}\ldots H_{i_{r+1}1} = \sum_{ij} H_{1i}(G^r)_{ij}H_{j1} \tag{1}$$

in which G, a member of a $(d-1)$-dimensional GOE, derives from H by putting to zero the first row and column, i.e. every $H_{1i}(=H_{i1})$. The B–W expansion§ is now

$$\begin{aligned} E &= \alpha + H_{11} + \sum_{r=0}^{\infty} \Lambda_r/E^{r+1} \\ &\simeq E_0 + (\partial E/\partial H_{11})_0 H_{11} + \sum_{r=0}^{\infty} (\partial E/\partial \Lambda_r)_0\, \delta\Lambda_r \\ &= \bar{E} + (\mathrm{d}\bar{E}/\mathrm{d}\alpha)\left\{H_{11} + \sum_{r=0}^{\infty} \delta\Lambda_r/\bar{E}^{r+1}\right\}. \end{aligned} \tag{2}$$

† Supported in part by the US Department of Energy.

‡ John Simon Guggenheim Memorial Fellow, 1977–8.

§ *Notation.* For any quantity A, $\bar{A}$ is its ensemble average. We have $\bar{H}_{ij} = 0$ and we take $\overline{H_{ij}^2} = (1+\delta_{ij})d^{-1}$, so that, for large d, the GOE spectrum has unit variance. We shall encounter ahead $\langle F\rangle \equiv d^{-1}\,\mathrm{Tr}(F)$ for any operator F. The symbol ($\simeq$) indicates equality to lowest order in d^{-1}, while the double arrow ($\Rightarrow$) will indicate the function generated by the quantities on its left as moments.

If, in the first form of (2), we use $\overline{(\Lambda_r/E^{r+1})} \simeq \bar{\Lambda}_r/(\bar{E})^{r+1}$, then $\bar{E}$ ($\simeq \alpha + \alpha^{-1}$) follows directly from (3, 4) below. For the variance of E we have, in the second form of (2), made a Taylor expansion of E about its value E_0 (no longer a random variable) defined by $H_{11} = 0$, $\delta\Lambda_r \equiv \Lambda_r - \bar{\Lambda}_r = 0$. In the last form we have used the fact that E is a function of $(\alpha + H_{11})$, not of either separately, so that $(\partial E/\partial H_{11})_0 = (\mathrm{d}E_0/\mathrm{d}\alpha)$; we have also used the identity that

$$(\partial f/\partial x)_y = -(\partial f/\partial y)_x(\partial y/\partial x)_f$$

for an arbitrary function $f(x, y)$, which gives

$$(\partial E/\partial \Lambda_r)_0 = (E_0)^{-r-1}(\mathrm{d}E_0/\mathrm{d}\alpha).$$

We have written the last form in terms of $\bar{E}$ rather than E_0 since, via the second form, ensemble averaging in the equation yields $\bar{E} \simeq E_0$.

Now, to evaluate the average and variance of E we need the averages and covariances of the Λ_r, which both follow from (1). Since G^r is independent of the H_{1i} we have $\bar{\Lambda}_{2\nu+1} = 0$ and

$$\bar{\Lambda}_{2\nu} = \sum_j \overline{(H_{1j})^2}\,\overline{(G^{2\nu})_{jj}} = \overline{\langle G^{2\nu}\rangle} \simeq \overline{\langle H^{2\nu}\rangle} = t_\nu = (\nu+1)^{-1}\binom{2\nu}{\nu}$$

$$\Rightarrow \overline{\rho_0}(x) = (2\pi)^{-1}(4-x^2)^{1/2} = \pi^{-1}\sin\psi(x) \tag{3}$$

where $\psi(x) = \cos^{-1}(-x/2)$, the angle between the negative x axis and the radius vector; thus $0 \leq \psi \leq \pi$. We have drawn here on Wigner's (1955) evaluation of the GOE moments as Catalan numbers (Riordan 1968), the moments of a zero-centered 'semicircle' of radius 2. Using a result for the Catalan generator

$$\sum_\nu t_\nu/x^{2\nu+1}$$

(Wigner 1955, Riordan 1968) we have from (2, 3) that, when $|\bar{E}| > 2$ (otherwise the series is divergent),

$$\bar{E} \simeq \alpha + \sum_\nu (t_\nu/(\bar{E})^{2\nu+1}) = \alpha + \tfrac{1}{2}[\bar{E} - \{\bar{E}^2 - 4\}^{1/2}] = \alpha + \alpha^{-1} \tag{4}$$

where the last form comes from solving the equation and is valid only when $|\alpha| > 1$, there being no solution for $|\alpha| < 1$. The result (4) agrees with the earlier derivations.

For the variance, using $\overline{H_{1i}H_{j1}H_{1k}H_{l1}} = d^{-2}\{\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\}$ when $(i, j, k, l) \neq 1$, we find immediately that

$$\overline{\delta\Lambda_{2\nu-r}\,\delta\Lambda_r} = \mathrm{cov}\{\langle G^{2\nu-r}\rangle, \langle G^r\rangle\} + 2d^{-1}\overline{\langle G^{2\nu}\rangle} \simeq 2d^{-1}t_\nu \tag{5}$$

in which we have dropped the covariance term which is of order d^{-2} (its value is given by Mon and French, 1975). Since the Λ_r are independent of H_{11} we have from the last form of (2), valid to lowest order in d^{-1}, that for the isolated eigenvalue

$$\begin{aligned}\mathrm{var}\,E &\simeq (\mathrm{d}\bar{E}/\mathrm{d}\alpha)^2\left\{2/d + \sum_{r,s=0}^{\infty} \overline{\delta\Lambda_r\,\delta\Lambda_s}/\bar{E}^{r+s+2}\right\} \\ &= \frac{2}{d}\left(\frac{\mathrm{d}\bar{E}}{\mathrm{d}\alpha}\right)^2\left\{1 + \sum_{\nu=0}^{\infty}\frac{(2\nu+1)t_\nu}{\bar{E}^{2\nu+2}}\right\} \\ &= \frac{2}{d}\left(\frac{\mathrm{d}\bar{E}}{\mathrm{d}\alpha}\right) = \frac{2}{d}\left(1 - \frac{1}{\alpha^2}\right)\end{aligned} \tag{6}$$

where in the second step we have carried out one summation by noting that $\overline{\delta\Lambda_r\,\delta\Lambda_s}$ depends only on $(r+s)$, and is zero when $r+s=$ odd and given by (5) when $r+s=$ even. In the third and fourth steps we have evaluated $1+\Sigma\,(2\nu+1)t_\nu/\bar{E}^{2\nu+2}=(\mathrm{d}\bar{E}/\mathrm{d}\alpha)^{-1}$ by taking an $\bar{E}$-derivative of equation (4). The variance, which vanishes in the asymptotic $-d$ limit, is small. It goes to zero (to order d^{-1}) as the level moves towards the semicircle $(\alpha\to 1_+)$ and increases to a maximum value $2/d$ as it moves away. This curious behaviour is not so surprising when we recall that the GOE levels have even smaller variances, $\sim d^{-2}\ln d$ (French *et al* 1978, Dyson 1962a).

For other states, first-order perturbation theory using G to split the degeneracy gives back the semicircle to order d^{-1}. When $|\alpha|<1$, treating the pairing matrix as perturbation, one finds, again to order d^{-1}, the unperturbed semicircle. These results are also in agreement with earlier derivations.

The method used here relies rather heavily on the simplicity of the K spectrum. For a more general method we turn to the principle referred to at the beginning and used by Wigner in evaluating the GOE moments. This is that, in a wide range of circumstances, the moment $\bar{M}_p=\langle\overline{H^p}\rangle$ is dominated by binary correlations so that $\bar{M}_{2\nu}/(\bar{M}_2)^\nu$ becomes the number of binary combinations of 2ν H's which contribute to the trace; the allowed combinations are only those in which any correlated pair contracts only around a fully pair-correlated set of H's; thus, with AA, BB denoting a correlated pair of H's $(=\Sigma\,H_{ij}H_{ji})$, $ABAB$ is forbidden while $AABB$ and $ABBA$ are allowed, the difference being that in the first the number of matrix-element contributions is down by order d^{-1} compared with the allowed ones. The same principle applies to the moments $\langle\overline{(H+\alpha K)^p}\rangle$; but since, for any operator Q, $\overline{AAQ}=(1+d^{-1})Q$ while $\overline{AQA}=d^{-1}(Q+\mathrm{Tr}\,Q)$, we see that, for large d, contractions around K are also forbidden unless we can take advantage of cyclic invariance of the trace, as we can with $\langle AKA\rangle$ but not with $\langle KAKA\rangle$. (In other words we should think of correlations on the circle rather than on the line.) Since $K^\zeta=K$ contractions around powers of K are similarly forbidden.

The counting theorem evaluates μ^p_ζ, the number of ways in which $(p-\zeta)$ pair-correlated H's can be inserted on a line which already contains ζ operators around which correlations are forbidden; thus with $p=4$, $\zeta=2$ we have H^2KK, KH^2K, KKH^2 and $HKKH$ so that $\mu^4_2=4$. Note that $(p-\zeta)$ is necessarily even. As indicated in (3) Wigner's result for $\zeta=0$ is $\mu^{2\nu}_0=t_\nu$; for $\zeta>0$ we have (Mon and French 1975, French *et al* 1978)

$$\mu^p_\zeta=\binom{p}{(p-\zeta)/2}\Rightarrow-\zeta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\{\bar{\rho}_0(x)v_{\zeta-1}(x)\}=(-1)^\zeta(2\pi\sin\psi)^{-1}\cos\zeta\psi \tag{7}$$

in which the functions $v_\zeta(x)=(-1)^\zeta(\sin\psi)^{-1}\sin(\zeta+1)\psi$ are the Chebyshev polynomials of the second kind, defined for $(-2,2)$ and orthonormal with the semicircular $\bar{\rho}_0(x)$, generated by the μ^p_0, as weight function. The result (7) was derived by considering the response to *infinitesimal* deformations of the GOE $(H\to H+\delta\alpha Q)$ and used to calculate GOE fluctuation measures in terms of the two-point correlation function (whose (p,q) moment is $2d^{-2}\,\Sigma\,\zeta\mu^p_\zeta\mu^q_\zeta$). The dominance of binary correlations is shown by the close agreement between these results and the exact ones, which extends in fact (Pandey 1978, to be published) to all three of the standard ensembles, orthogonal, unitary and symplectic (Dyson 1962b), being exact for the unitary ensemble.

We see now that the ensemble-averaged p'th moment of H_α becomes

$$\bar{M}_p(\alpha)\equiv\langle\overline{(H_\alpha)^p}\rangle=\sum_{\zeta=0}^{p}\mu^p_\zeta\langle(\alpha K)^\zeta\rangle=\bar{M}_p+d^{-1}\sum_{\zeta=1}^{p}\mu^p_\zeta\alpha^\zeta$$

$$\xrightarrow{|\alpha|<1} \bar{\rho}_\alpha(x) = \bar{\rho}_0(x) + (2\pi d \sin\psi)^{-1} \sum_{\zeta\geq 1} (-\alpha)^\zeta \cos\zeta\psi$$

$$= \bar{\rho}_0(x) - (2\pi d \sin\psi)^{-1} \frac{(\cos\psi + \alpha)}{(\alpha + \alpha^{-1} + 2\cos\psi)}$$

$$= \bar{\rho}_0(x) + \frac{1}{2\pi d} \frac{x - 2\alpha}{(\alpha + \alpha^{-1} - x)(4 - x^2)^{1/2}}$$

$$\simeq \bar{\rho}_0(x) \tag{8}$$

so that when $|\alpha| < 1$, H_α has, to order d^{-1}, simply the semicircular spectrum. It should not be taken for granted however that the d^{-1} correction displayed in (8), which of course vanishes outside the semicircle, is the only correction of that order.

We have in (8) restricted ourselves to $|\alpha| < 1$. For $|\alpha| > 1$, for which the final ζ summation is not convergent, we should in principle evaluate the ζ summation before inverting the moment expansion. But more simply we recognise that, in the moments (8), the ζ sum is a truncated binomial expansion and then by changing variables we relate the densities for reciprocal values of α. Explicitly, writing

$$X(\alpha) = \sum_{\zeta=1}^{p} \mu_\zeta^p \alpha^\zeta$$

and noting that $\mu_\zeta^p = \mu_{-\zeta}^p$ we find

$$X(\alpha) = (\alpha + \alpha^{-1})^p - \binom{p}{p/2} - X(1/\alpha),$$

the second term being zero when p is odd. Then from (8) we have, observing that

$$\binom{p}{p/2} \Rightarrow -2\bar{\rho}_0'(x)/x,$$

$$\bar{M}_p(|\alpha| > 1) \Rightarrow \bar{\rho}_\alpha(x) = \bar{\rho}_0(x) + d^{-1}\Big\{\delta(x - \alpha - \alpha^{-1})$$

$$- \frac{1}{\pi[4 - x^2]^{1/2}} - \frac{1}{2\pi} \frac{x - 2/\alpha}{(\alpha + \alpha^{-1} - x)[4 - x^2]^{1/2}}\Big\}$$

$$= \bar{\rho}_0(x) + \frac{1}{2\pi d} \frac{x - 2\alpha}{(\alpha + \alpha^{-1} - x)[4 - x^2]^{1/2}} + \frac{1}{d}\delta(x - \alpha - \alpha^{-1})$$

$$\simeq \bar{\rho}_0(x) + d^{-1}\delta(x - \alpha - \alpha^{-1}) \tag{9}$$

and have thus rederived the earlier results along with d^{-1} corrections to the semicircle, as given already by Jones *et al* (1978). The variance of the isolated eigenvalues, which we have given above, can also be derived by calculating $\bar{M}_p(\alpha)$ in (8) to d^{-2}-order terms, or by calculating the covariance of the moments $\langle (H_\alpha)^p \rangle$; we have verified (6) by the latter procedure.

The spectrum and fluctuations for more general Hamiltonians are of interest in studying the effects which an almost good symmetry produces (by moderating the level repulsion) on the energy level and strength fluctuations (Dyson 1962c). This problem is solvable in terms of the density and correlation functions for a GOE augmented by a multiple of the bilinear Casimir operator for the group in question (K is such an operator, for the symplectic group, but one of little interest in the present context). A similar problem, which can be handled in a similar way, arises for a system with a strongly collective normal mode in which the other parts of the Hamiltonian are representable by a random operator.

Formally we can apply the same procedure as above to the Hamiltonian $(H+\alpha Q)$. Starting once again with the first expansion of $\tilde{M}_p(\alpha)$ given by (8) we find

$$\overline{\rho_\alpha}(x)=\overline{\rho_0}(x)+\frac{1}{2\pi}(4-x^2)^{-1/2}\langle(x-2\alpha Q)/(\alpha Q+(\alpha Q)^{-1}-x)\rangle$$
$$+d^{-1}\sum_i{}'\,\delta(x-\alpha q_i-(\alpha q_i)^{-1})$$
$$=\bar{\rho}_0(x)-\frac{1}{\pi}\operatorname{Im}\int\frac{2\alpha y f(y)\,\mathrm{d}y}{[x+(x^2-4)^{1/2}-2\alpha y](x^2-4)^{1/2}} \quad (10)$$

a slight generalisation of a form given by Jones *et al* (1978). Here the q_i are the eigenvalues of Q and the summation extends over all i for which $|\alpha q_i|>1$. It should be clear that the trace, involving a sum over eigenvalues, exists even if Q has zero eigenvalues. In the last form $f(y)$ is the Q-eigenvalue density, and, in the integral, x is endowed with a small positive imaginary part.

The result (10) is not valid for an arbitrary Q formally because, unlike the situation with pairing, contractions around powers of Q may not be inhibited even if contractions around Q are. A sufficient condition for validity is that $\langle Q^p\rangle \gg \Pi_i\langle Q^{p_i}\rangle$ where the p_i form any partition of p into two or more parts. For general Q the density is easily solvable for large and small $|\alpha|$; for arbitrary α, which we need for the problems mentioned above, the combinatorial problems are not yet solved but appear to be solvable, at least for the density.†

The statistical independence of the matrix elements of a GOE realised in a many-particle space implies simultaneous interactions between all particles. A generalisation, free of this defect, is the embedded GOE (EGOE) of k-body interactions, formed by a GOE generated in a k-particle space but acting in an $(m>k)$-particle space. Once again binary correlations are dominant for the density but as we increase m (for fixed k) the inhibitions on the allowed correlations gradually disappear (because different operators in a trace act on different particles and therefore *effectively commute in the large-m limit*). The EGOE density then gradually changes from semicircular to Gaussian (Mon and French 1975) while that for the augmented EGOE with Hamiltonian $(H+\alpha Q)$ becomes a convolution of the Gaussian density $\rho_{\mathscr{G}}^{(H)}(x)$ with the density (spectrum), $\rho^{(Q)}(x)$, of αQ. Explicitly, writing $\langle F\rangle^m$, in analogy with $\langle F\rangle$, for the m-particle average, and considering only $m\gg k$

$$\overline{\langle H^{2\nu}\rangle^m}=(2\nu-1)!!\{\overline{\langle H^2\rangle^m}\}^\nu\Rightarrow\rho_{\mathscr{G}}^{(H)}(x)$$
$$\times\overline{\langle (H+\alpha Q)^p\rangle^m}$$
$$\simeq\sum_{r=0}^{p}\binom{p}{r}\alpha^r\overline{\langle Q^rH^{p-r}\rangle^m}=\sum_{r=0}^{p}\binom{p}{r}\langle(\alpha Q)^r\rangle^m\overline{\langle H^{p-r}\rangle^m}$$
$$\Rightarrow\rho_{\mathscr{G}}^{(H)}\otimes\rho^{(Q)}[x] \quad (11)$$

in which, for this case, the corrections implied by $(\simeq)$ are in inverse powers of particle number. The centroid of $\rho_{\mathscr{G}}^{(H)}(x)$ is seen to be zero and its variance $\binom{m}{k}$. For large $|\alpha|$

† *Note added in proof.* This has now been done.

the Q spectrum is then preserved but the levels take on a Gaussian spread about their unperturbed positions. For small $|\alpha|$ the Q spectrum is lost in the background.

The EGOE long-range fluctuations are similarly dominated by binary correlations; by the action of a central limit theorem they die out extremely rapidly as particle number increases. The short-range fluctuations however ($\lambda \sim$ level spacing), which are *not* dominated by binary correlations, escape the action of the CLT; the consequence is a very sharp separation of the secular and fluctuation behaviour.

The GOE is a strongly ergodic ensemble (Pandey 1978) so that essentially all of its members display the same properties. For the EGOE this result applies, at least for the eigenvalue density (Mon and French 1975), so that we should expect Gaussian spectra for individual Hamiltonians when $m \gg k$. On the other hand operators with special algebraic properties are found with negligible weight in the GOE and may therefore have a different asymptotic (large-m) spectrum. Thus for example (French and Draayer 1978) the bilinear Casimir operator for an l-dimensional Lie group, realised in an m-particle space as a subgroup of $U(N)$ (the group of unitary transformations among the N underlying single-particle states), has for large m a χ_l^2 distribution independently of the specific way in which the algebra is realised. More generally than this (Halemane 1978, private communication) one finds the asymptotic spectrum for an arbitrary two-body operator essentially in terms of a convolution of a number of χ_1^2 densities, one for each 'component' of the Hamiltonian; just as χ_l^2 becomes Gaussian for large l so would the spectra for 'most' H's. In these cases also, which do not involve ensemble averaging, the results are determined by the dominance of binary correlations.

References

Dyson F J 1962a *J. Math Phys.* **3** 166–75
—— 1962b *J. Math. Phys.* **3** 1199–215
—— 1962c *J. Math. Phys.* **3** 1191–8
Edwards S F and Jones R C 1976 *J. Phys. A: Math. Gen.* **9** 1595–1603
French J B and Draayer J P 1978 *Proc. of the 1978 Int. Conf. on Group Theory and Mathematical Physics* ed W Beiglböck (Berlin: Springer) in press
French J B, Mello P A and Pandey A 1978 *Ann. Phys., NY* **113** 277–93
Jones R C, Kosterlitz J M and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** L45–8
Mon K K and French J B 1975 *Ann. Phys., NY* **95** 90–111
Pandey A 1978 *Ann Phys., NY* in press
Riordan J 1968 *Combinatorial Identities* (New York: Wiley)
Wigner E P 1955 *Ann. Math.* **62** 548–64